Monthly Forecasting of GDP with Mixed-Frequency Multivariate Singular Spectrum Analysis[1]

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Abstract
The literature on mixed-frequency models is relatively recent and has found applications across economics and finance. The standard application in economics considers the use of (usually) monthly variables (e.g. industrial production) for predicting/fitting quarterly variables (e.g. real GDP). This paper proposes a multivariate singular spectrum analysis (MSSA) based method for mixed-frequency interpolation and forecasting, which can be used for any mixed-frequency combination. The novelty of the proposed approach rests on the grounds of simplicity within the MSSA framework. We present our method using a combination of monthly and quarterly series and apply MSSA decomposition and reconstruction to obtain monthly estimates and forecasts for the quarterly series. Our empirical application shows that the suggested approach works well, as it offers forecasting improvements on a dataset of eleven developed countries over the last 50 years. The implications for mixed-frequency modelling and forecasting, and useful extensions of this method, are also discussed.

JEL: C18, C53, E17

Keywords: multivariate SSA, mixed-frequency, GDP, industrial production, forecasting.

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1. Introduction

The last two decades has seen the development of an expanding body of literature on the theoretical foundations and empirical applications of singular spectrum analysis (SSA) (see for example Kalantari, Yarmohammadi, Hassani, & Silva, 2018; Hassani, Silva, Gupta, & Das, 2018; Silva, Ghodsi, Ghodsi, Heravi, & Hassani, 2017; and references therein). SSA turns out to be a fairly versatile approach for modelling and forecasting, as it allows users to cope with both linear and nonlinear, as well as with stationary and non-stationary, time series structures (Sanei & Hassani, 2015). Although SSA is a powerful tool in time series analysis and has been applied in a wide range of fields, there has been an increasing interest over recent years in the use of SSA in economics and finance (see for example Hassani & Thomakos, 2010; and Hassani & Patterson, 2014; for two comprehensive overviews).

SSA-based models have proved to be reasonably useful for forecasting economic and financial variables. For instance, SSA has been used to forecast economic activity, namely industrial production (Hassani, Heravi, & Zhigljavsky, 2009; Hassani, Heravi, & Zhigljavsky, 2013; Hassani, Ghodsi, Silva, & Heravi, 2016; Patterson, Hassani, Heravi, & Zhigljavsky, 2011; Silva, Hassani, & Heravi, 2018), and GDP (Hassani & Zhigljavsky, 2009; Hassani, Soofi, & Avazalipour, 2011; Hassani, Heravi, Brown, & Ayoubkhani (2013). Likewise, Papailias & Thomakos (2017) consider a set of US variables including GDP. In addition, de Carvalho, Rodrigues, & Rua (2012) and de Carvalho & Rua (2017) resort to SSA for nowcasting the US output gap, a first application of the ideas for mixed-frequency SSA. SSA has also been used for forecasting tourism, and in particular UK tourism income by Beneki, Eeckels, & Leon (2012), US tourist arrivals by Hassani, Webster, Silva, & Heravi (2015), and European tourist arrivals by Hassani, Silva, Antonakakis, Filis, & Gupta (2017), and Silva, Hassani, Heravi, & Huang (2019). Applications exploiting SSA for forecasting exchange rates (Lisi & Medio, 1997; and Hassani, Soofi, & Zhigljavsky, 2009), inflation (Hassani, Soofi, & Zhigljavsky, 2013; Silva, Hassani, & Otero, 2018), energy (Beneki & Silva, 2013; Silva, 2013), and more recently, fashion consumer behaviour (Silva, Hassani, Madsen & Gee, 2019) are also available.

This paper attempts to solve the mixed-frequency interpolation/forecasting problem in the context of SSA. Such an approach has not been considered in the previous literature and has certain *a priori* advantages which suggest the advisability of examining its efficacy: first, SSA is a complete, stand-alone smoothing/filtering/forecasting method; second, SSA is model-free and its performances across different types of time series have been found to be very good (Ghodsi, Hassani, Rahmani, & Silva, 2017); and, third, SSA can easily handle multivariate applications.

Early works on the use of mixed frequencies were related to the strand of literature that focuses on the temporal disaggregation of time series, namely by obtaining high-frequency estimates of a series observed at a lower frequency by resorting to high-frequency indicators. Naturally, the underlying relationship can be used to obtain estimates for the out-of-sample period covered by the low-frequency variable. In this respect, one should mention the well-known seminal work of Chow and Lin (1971), who developed a regression-based framework for temporal disaggregation. However, the usefulness of the proposed method was limited in practice, as it depends on the validity of the regression model, which is bound by a variety of parametric assumptions that are unlikely to hold in the real world. Subsequent work includes the studies by Fernandez (1981), Litterman (1983), Wei and Stram (1990), Guerrero (1990), Liu and Hall (2001) and Santos Silva and Cardoso (2001), among others. Recently, the interest in mixed-frequency models has increased again with, for example, the MIDAS (mixed-data sampling) approach of Eric Ghysels and co-authors (see for example Ghysels, Sinko, & Valkanov, 2007; and Andreou, Ghysels, & Kourtellos, 2010, 2013). The body of work on the use of the MIDAS approach for forecasting includes the studies by Kuzin, Marcellino, and Schumacher (2011), Clements and Galvão (2012), Monteforte and Moretti (2013), Galvão (2013), Ghysels and Ozkan (2015), Duarte, Rodrigues, and Rua (2017), among others. In addition, dynamic factor models for mixed-frequency forecasting have also been developed (see for example Schumacher & Breitung, 2008; Mariano & Murasawa, 2003; Foroni & Marcellino, 2014; Marcellino, Porqueddu, & Venditti, 2016). Other mixed-frequency approaches have been pursued by, for example, Gotz, Hecq, and Urbain (2014), Carriero, Clark, and Marcellino (2015), Schorfheide and Song (2015), Foroni, Guérin, and Marcellino (2015), Barsoum and Stankiewicz (2015), and Marcellino and Sivec (2016).

All of the above approaches, and their refinements, not only attract a considerable amount of attention in the literature, but also offer solutions for the faster tracking of important economic variables, mainly GDP, by approximating their paths with the use of auxiliary, correlated variables that are observed at higher frequencies (such as monthly, weekly or even daily). Based on the literature mentioned above, it is clear that mixed-frequency forecasting is an important topic. However, the majority of the models used for mixed-frequency forecasting are restricted by their parametric nature and related assumptions pertaining to normality, linearity and stationarity. Here, we extend the list of models used for mixed-frequency forecasting by resorting to the multivariate singular spectrum analysis (MSSA). Whilst initially SSA and MSSA were used for nowcasting through the work done by de Carvalho and Rua (2017), this paper marks the introduction of the use of MSSA for mixed-frequency forecasting. Given its nonparametric nature, modelling with MSSA enables users to ensure that there is no loss of information, as no data transformations are required, and it enables the smoothing, filtering and signal extraction that can also be useful when modelling data with mixed frequencies.

We follow the same underlying intuition as the other methods in the literature when developing our approach in the simplest possible context: interpolating and forecasting the monthly path of GDP, which is observed only at the quarterly frequency, by using industrial production as a highly correlated monthly proxy. We highlight the usefulness of our suggested approach, based on mixed-frequency multivariate SSA, by considering an empirical application that uses data for eleven developed countries for the period from the beginning of 1960 up to the end of 2013. We find that, once a monthly measure of GDP growth has been obtained through the mixed frequency multivariate SSA, one can improve the forecasting performance substantially by taking into account the monthly dynamics vis-à-vis the case where one forecasts the quarterly series. The forecasting gains are noteworthy across all countries, with the average improvement in terms of both the root mean squared forecast error (RMSFE) and the mean absolute forecast error (MAFE) being around 40%. We also find that the suggested method still improves the quarterly counterpart even in a pseudo real-time environment, though only slightly. Moreover, one should stress the fact that this approach allows economic developments to be tracked on a monthly basis, which is valuable *per se* for real time monitoring and policymaking.

The remainder of the paper is organized as follows. Section 2 introduces our novel approach based on multivariate SSA for coping with a mixed frequency data framework. Section 3 describes the dataset considered. Section 4 conducts the empirical application and discusses the results. Finally, Section 5 concludes.

2. A mixed-frequency multivariate SSA approach

2.1. Multivariate SSA: decomposition and reconstruction

This section reviews (one of the possible ways of doing) multivariate SSA. Our presentation is for the bivariate case, but pure multivariate adaptations are straightforward. Thus, consider the bivariate time series $\left\{\mathbf{X}_t \stackrel{\text{def}}{=} [X_{t1}, X_{t2}]^\top\right\}_{t\in\mathbb{S}}$, which takes values in $\mathcal{R}_X \subseteq \mathbb{R}$. The index set $\mathbb{S}$ can be either $\mathbb{Z}$ or $\mathbb{N}$, thus covering the cases of both stationary and nonstationary time series. It is assumed that both series are already scaled appropriately and expressed in commensurable units of measurement. This becomes important in the next subsection, when we introduce mixed frequencies.

Suppose that we have a sample of size N available, and let m denote the embedding dimension that we propose to use. Applying the hankelization operator $\mathcal{H}_m(\cdot)$ to each of the component series of $\mathbf{X}_t$, we obtain the trajectory $(n \times m)$ matrices $\mathbf{T}_{i,} \stackrel{\text{def}}{=} \mathcal{H}_m(X_{1i}, X_{2i}, \ldots, X_{Ni})$, for $i = 1, 2$ and $n = N - m + 1$. Concatenating the trajectory matrices horizontally, we obtain the $(n \times 2m)$ MSSA trajectory matrix $\mathbf{T}_X \stackrel{\text{def}}{=} [\mathbf{T}_1, \mathbf{T}_2]$ that we use for decomposition and reconstruction.

The $(2m \times 2m)$ sample covariance matrix is then defined as:

$$\mathbf{C} \stackrel{\text{def}}{=} n^{-1}\mathbf{T}_X^\top \mathbf{T}_X, \tag{1}$$

which is block-symmetric and is given by:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_2 \end{bmatrix}, \tag{2}$$

such that it contains the own and cross-covariance matrices as its elements. Denote the spectral decomposition of $\mathbf{C}$, in standard notation, by:

$$\mathbf{C}_n \stackrel{\text{def}}{=} \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\top = \sum_{j=1}^{2m} \lambda_j \mathbf{v}_j \mathbf{v}_j^\top, \tag{3}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2m}$. Splitting the $(2m \times 2m)$ matrix $\mathbf{V}$ that holds the eigenvectors appropriately into $\mathbf{V} \stackrel{\text{def}}{=} \left[\mathbf{V}_1^\top, \mathbf{V}_2^\top\right]^\top$, we can estimate the individual trajectory matrices as:

$$\widehat{\mathbf{T}}_i(k) \stackrel{\text{def}}{=} \mathbf{T}_i \mathbf{Q}(k), \tag{4}$$

where $\mathbf{Q}(k) \stackrel{\text{def}}{=} \sum_{j \in \mathcal{I}_k} \mathbf{v}_{ij} \mathbf{v}_{ij}^\top$ and $\mathbf{V}_i \stackrel{\text{def}}{=} [\mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots, \mathbf{v}_{im}]$ for $i = 1, 2$. Here, $k \stackrel{\text{def}}{=} \mathrm{d}im\mathcal{I}_k \leq m$ and $\mathcal{I}_k$ denotes the set of eigenvalue indices used in the reconstruction process. Finally, we obtain the reconstructed series by applying the diagonal averaging operator $\mathcal{D}_{(m,N)}(\cdot)$ to the estimated trajectory matrix as in:

$$\left\{\widehat{X}_{ti}(k)\right\}_{t=1}^{N} \stackrel{\text{def}}{=} \mathcal{D}_{(m,N)}\left[\widehat{\mathbf{T}}_i(k)\right]. \tag{5}$$

2.2. MSSA when one series has a higher frequency

We now turn to the proposed methodology for MSSA when one of the series exhibits a higher sampling frequency, i.e., mixed-frequency MSSA or MFMSSA. We make things concrete and relate them to our empirical results by supposing that X_{t1} is a quarterly time series and X_{t2} is a monthly time series. Now, N denotes the length of the higher-frequency series; note that we must have expressed the units of measurement of the two series so that they match the higher sampling frequency. For example, when dealing with a quarterly and a monthly series, which are both expressed as growth rates, we must transform them so that their growth rates correspond to the same period and the same sampling interval. In particular, our empirical application takes the quarterly growth rate of GDP to be the value observed at the third month of the quarter, whereas the values for the remaining months of the quarter are unknown. In the case of industrial production, the value for each month corresponds to the quarter-on-quarter growth rate for the quarter ending at that month. Hence, as a first step, we prepare our data according to the following format, marking the positions of the quarterly series for which actual data are available:

$$\begin{bmatrix} (Q1,M1) & X_{11} & X_{21} & 0 \\ (Q1,M2) & X_{11} & X_{22} & 0 \\ (Q1,M3) & \underline{X_{11}} & X_{23} & 1 \\ (Q2,M4) & X_{21} & X_{24} & 0 \\ (Q2,M5) & X_{21} & X_{25} & 0 \\ (Q2,M6) & \underline{X_{21}} & X_{26} & 1 \\ (Q3,M7) & X_{31} & X_{27} & 0 \\ (Q3,M8) & X_{31} & X_{28} & 0 \\ (Q3,M9) & \underline{X_{31}} & X_{29} & 1 \\ (Q4,M10) & X_{41} & X_{2,10} & 0 \\ (Q4,M11) & X_{41} & X_{2,11} & 0 \\ (Q4,M12) & \underline{X_{41}} & X_{2,12} & 1 \end{bmatrix}, \tag{6}$$

so that initially we fill-in the monthly values of the quarterly series with the actual, end-of-quarter, values. Note that we mark the positions/dates of the quarterly series that should not be changed during our iterations below with ones in the last column. That is, the positions for which actual data are available must be retained as such, but the data for the other monthly positions for which we have no information can be approximated.[1]

The idea that we use for the monthly interpolation is very simple: pass the initial data to the standard MSSA approach described earlier and obtain the fitted values, re-insert the actual values in the fixed positions indicated by the ones in Eq. (6), measure the mean-squared deviations between rounds of approximation, and terminate when an appropriate condition is met. If we denote the value of the slower-frequency series at the r^{th} round/iteration of the above procedure by $X_{t1}^{(r)}$, we can illustrate the method schematically as follows:

Step 0. Using the data formatting in Eq. (6), apply MSSA to the two series and obtain the higher-frequency fitted values $\widehat{X}_{t1}^{(0)}$. Initialize the mean-squared measure as $RMSE(0) \stackrel{\text{def}}{=} \sqrt{N^{-1} \sum_{t=1}^{N} \left[\widehat{X}_{t1}^{(0)}\right]^2}$.

Step 1. Substitute the actual values into $\widehat{X}_{t1}^{(0)}$ in the positions indicated by the ones in Eq. (6) and reapply MSSA to obtain the updated higher-frequency fitted values $\widehat{X}_{t1}^{(1)}$; again, substitute the actual values as before into this

1. Our objective here is to find an unknown high-frequency series whose last values are consistent with a known low-frequency series; alternatively, though, depending on the application, one could search for a series that could be considered to be consistent with the first values, sums or averages.

new series and compute the new mean-squared measure as $RMSE(1) \stackrel{\text{def}}{=} \sqrt{N^{-1}\sum_{t=1}^{N}\left[\widehat{X}_{t1}^{(0)} - \widehat{X}_{t1}^{(1)}\right]^2}$.

Step 2. If $RMSE(1) \leq RMSE(0)$ but $|RMSE(1) - RMSE(0)| > \varepsilon$, for some small predefined number ε, then iterate again by passing $\widehat{X}_{t1}^{(1)}$ to another round of MSSA to obtain $\widehat{X}_{t1}^{(r)}$ for $r > 1$.

Step 3. While $RMSE(r+1) \leq RMSE(r)$ but $|RMSE(r+1) - RMSE(r)| > \varepsilon$, continue along steps 1 and 2; if $RMSE(r+1) \leq RMSE(r)$ and $|RMSE(r+1) - RMSE(r)| \leq \varepsilon$ terminate the iterations and use $\widehat{X}_{t1}^{(r+1)}$ as the final estimate of the higher-frequency series.

Note that the actual, appropriately measured, quarterly series are always used in the positions with ones in Eq. (6). Thus, only the missing months are filled-in with the interpolated values. In of all our experiments, we set ε to the range $[1e-5, 5e-4]$ and achieved convergence of the above algorithm in an average of fewer than 20 iterations.

There are a number of practical issues that the reader will no doubt ask about immediately: how should one select the higher-frequency series so as to achieve meaningful interpolation values for the lower-frequency series? What kinds of embedding dimensions work well for this procedure, assuming that the first question is answered? How many eigenvectors should one use in the reconstruction phase of the higher-frequency fitted values of the lower-frequency series? These questions must be answered before one can implement the method. Here are our suggestions.

It is clear that the higher-frequency series needs to be highly correlated if there is to be any a priori reason to believe that a good interpolation will take place. Therefore, we start by seeking good (i.e., highly correlated) higher-frequency proxies for the lower-frequency series. If we agree on this suggestion, then one can consider doing a direct search on the length of the embedding dimension m that maximizes the statistical correlation between the actual higher-frequency series X_{t2} and the fitted values of the lower-frequency series $\widehat{X}_{t1}^{(r)}$. That is, a plausible value m^* for m in our interpolation procedure might be selected to satisfy the condition:

$$m^* \stackrel{\text{def}}{=} \text{argmax}_m \mathsf{Corr}\left(X_{t2}, \widehat{X}_{t1}^{(r)}\right). \tag{7}$$

Our empirical application below found that selecting a value of m that was equal to the higher sampling frequency (i.e., $m = 12$) was a reasonable compromise, as it gave results that were practically identical to a search for m^* as above. In essence, we found that the correlation was maximized at or close to the higher sampling frequency. Such results merit further investigation with other kinds of series as well.

Finally, we answered the last question by again looking at the ex-post correlation between X_{t2} and $\widehat{X}_{t1}^{(r)}$. We found that essentially all eigenvectors

	Min	Max	Mean	Med	IQR	SD	CV	Skew	KS (p)	ADF	r
Quarterly GDP											
Austria	−4.38	4.51	0.72	0.73	1.16	1.04	144.93	−0.36	0.07†	−5.97	0.52*
Belgium	−2.16	3.76	0.67	0.73	0.89	0.74	110.42	−0.36	0.02†	−4.58	0.49*
France	−7.58	11.37	0.70	0.65	0.85	1.17	168.15	1.99	<0.01	−6.64	0.90*
Germany	−4.48	4.48	0.62	0.66	1.26	1.12	180.67	−0.34	0.03†	−7.45	0.66*
Italy	−2.88	6.00	0.63	0.56	1.29	1.04	166.46	0.53	0.03†	−5.28	0.75*
Japan	−3.97	5.7	0.99	0.91	1.60	1.34	135.12	−0.08	0.04†	−4.50	0.63*
Netherlands	−6.27	8.94	0.69	0.76	1.13	1.45	209.24	0.27	<0.01	−8.14	0.38*
Portugal	−2.33	4.90	0.81	0.81	1.46	1.22	151.38	0.16	0.20†	−4.31	0.58*
Sweden	−4.77	5.50	0.64	0.66	1.33	1.31	204.78	−0.18	<0.01	−7.48	0.63*
United Kingdom	−2.41	5.28	0.62	0.65	0.84	0.97	157.89	0.43	<0.01	−6.01	0.67*
United States	−2.11	3.89	0.76	0.77	0.82	0.85	111.01	−0.27	<0.01	−4.34	0.76*
Monthly IP											
Austria	−6.44	7.22	0.92	1.04	2.23	1.75	189.93	−0.39	0.03†	−9.71	
Belgium	−10.36	17.16	0.68	0.75	2.12	2.20	323.50	0.22	<0.01	−13.87	
France	−14.12	21.73	0.47	0.45	1.85	2.27	486.28	1.36	<0.01	−15.99	
Denmark	−13.40	5.89	0.61	0.76	2.27	2.00	325.63	−1.73	<0.01	−9.62	
Italy	−10.85	15.24	0.54	0.56	2.49	2.47	459.33	−0.10	<0.01	−11.44	
Japan	−19.70	10.42	1.01	1.13	2.48	2.81	277.49	−1.99	<0.01	−9.53	
Netherlands	−7.30	9.82	0.70	0.77	2.69	2.01	285.62	−0.25	0.09†	−12.30	
Portugal	−7.51	10.86	0.77	0.70	2.83	2.49	325.5	0.41	<0.01	−13.18	
Sweden	−11.47	7.16	0.61	0.64	2.40	2.19	356.96	−0.79	<0.01	−12.23	
United Kingdom	−7.04	8.64	0.26	0.30	1.53	1.60	610.21	−0.03	<0.01	−13.61	
United States	−7.38	4.51	0.69	0.90	1.41	1.58	228.36	−1.27	<0.01	−9.35	

Note: † indicates that the data are distributed normally based on the Kolmogorov-Smirnov (KS) test at a p-value of 0.01. r indicates the Pearson's correlation between GDP and IP. * indicates a statistically significant correlation between GDP and IP quarterly growth at the usual 0.05 significance level.

TABLE 1. Descriptive statistics for GDP and IP growth rates.

should be used in the reconstruction phase of MSSA; that is, all available statistical information from the decomposition of the joint trajectory matrix. This makes intuitive sense, as there is no reason to believe that one can discard information from the higher frequency when obtaining interpolated values for the lower frequency, and thus we consistently found that k should be set equal to m.

3. Data

The data used in our empirical application include the quarterly real GDP from Q1:1960 up until Q4:2013 and the monthly industrial production from January 1960 to December 2013 for 11 developed countries, namely Austria, Belgium, France, Denmark, Italy, Japan, Netherlands, Portugal, Sweden, United Kingdom and United States. The data have been retrieved from the OECD Main Economic Indicators database. The data are referred to as mixed-frequency because the GDP figures are quarterly whilst the IP figures are reported on a monthly basis. In general, the GDP growth and IP appear to be highly correlated for all countries, and this study exploits this dependence in order to improve the forecastability of GDP. Table 1 provides a set of descriptive statistics regarding GDP and IP growth over the sample period considered.

Our analysis begins by considering the quarterly GDP growth. The minimum and maximum columns indicate that, during this period, both the worst and the best quarterly growth rates achieved during a particular quarter were those reported for France. In terms of the average quarterly GDP growth, the highest average has been reported by Japan, whilst the lowest average quarterly GDP growth rate is equivalent in Germany and the UK. Based on the standard deviation, the most variable quarterly GDP growth has been recorded for the Netherlands, while the most stable quarterly growth rate for GDP was recorded for the US. The KS test for normality indicates evidence of normally distributed quarterly GDP growth for six of these countries, whilst all quarterly GDP growth series are found to be stationary based on an augmented Dickey-Fuller (ADF) test for a p-value of 0.05. All correlations between GDP and IP growth were found to be statistically significant at the 5% significance level.

We can perform a similar analysis in terms of IP. The worst and best monthly IP growth rates during this time period were reported for Japan and France, respectively. Japan also has the highest average monthly IP growth rate, whilst the UK reports the lowest average growth. The KS test for normality indicates that the majority of the monthly IP series are skewed, and therefore the results remain unchanged if we give prominence to the median growth rather than the mean growth, with the highest median monthly IP growth being reported for Japan and the lowest for the UK. The SD criterion suggests that Japan has the least stable average monthly IP growth rate and that the US has the most stable. If we consider the IQR, as most of the data are skewed, then the least stable average IP growth is recorded for Portugal, whilst the US continues to report the most stable average IP growth rate. As with quarterly GDP growth rates, there is no evidence in monthly IP growth rates of the time series being nonstationary based on the ADF test for unit root problems. The coefficient of variation (CV) criterion enables us to compare the variation between the quarterly GDP growth and the monthly IP growth. The CV clearly indicates that the monthly IP growth rate is more variable than the quarterly GDP growth for all countries.

We then go a step further and perform an ANOVA test to determine whether there are statistically significant differences between countries in terms of their quarterly GDP growth rates and monthly IP growth rates. Interestingly, the post-hoc Tukey HSD ($p = 0.05$) test shows evidence of statistically significant differences between the average quarterly GDP growth rates of only three combination of countries, namely Germany and Japan, Italy and Japan, and Japan and the UK. On the other hand, the post-hoc Tukey HSD test ($p = 0.05$) for statistically significant differences in the average monthly growth rates of IP found the following significant combinations: Austria and France, Austria and UK, Belgium and UK, France and Japan, Germany and Japan, Italy and Japan, Japan and Sweden, Japan and UK, Netherlands and

UK, Portugal and UK, and UK and US, all of which were found to be countries with statistically significant differences in monthly IP growth rates.

4. Evaluating the MFMSSA approach

As was discussed in Section 2.2, we conduct the following exercise here: based on the whole sample period, we take the quarterly GDP growth and perform the MFMSSA interpolation, using the monthly industrial production growth as the covariate. The resulting monthly estimates for GDP growth are displayed in Figure 1 alongside the corresponding covariate series. It can be seen from Figure 1 that the monthly estimates of GDP are much less volatile than those for industrial production, and Table 2 reports the corresponding correlations.[2]

The weighted correlation (w-correlation) is another measure of the dependence between two series that is considered when evaluating the appropriateness of the separability between signal and noise, as achieved via MSSA. If we consider two time series Y_N^1 and Y_N^2, then the w-correlation can be calculated as:

$$\rho_{12}^{(w)} = \frac{\left(Y_N^{(1)}, Y_N^{(2)}\right)_w}{\| Y_N^{(1)} \|_w \| Y_N^{(2)} \|_w},$$

where $Y_N^{(1)}$ and $Y_N^{(2)}$ are two time series, $\| Y_N^{(i)} \|_w = \sqrt{\left(Y_N^{(i)}, Y_N^{(i)}\right)_w}$, $\left(Y_N^{(i)}, Y_N^{(j)}\right)_w = \sum_{k=1}^{N} w_k y_k^{(i)} y_k^{(j)} \quad (i,j = 1,2)$, $w_k = \min\{k, L, N-k\}$ (here, assume $L \leq N/2$).

As was explained by Hassani, Mahmoudvand, Zokaei, and Ghodsi (2012), if the absolute value of the w-correlations is small, then the corresponding series are almost w-orthogonal, but if it is large, then the two series are far from being w-orthogonal and are therefore badly separable. As an example, Figure 2 presents the w-correlation matrix for the United States, to show the dependence between the signal and noise components.

2. We also find that the IP series does not Granger-cause the monthly GDP series, which seems natural, as the information from the IP is already embedded in the estimation of the monthly GDP, but the monthly GDP series seems to Granger-cause the IP series in most countries.

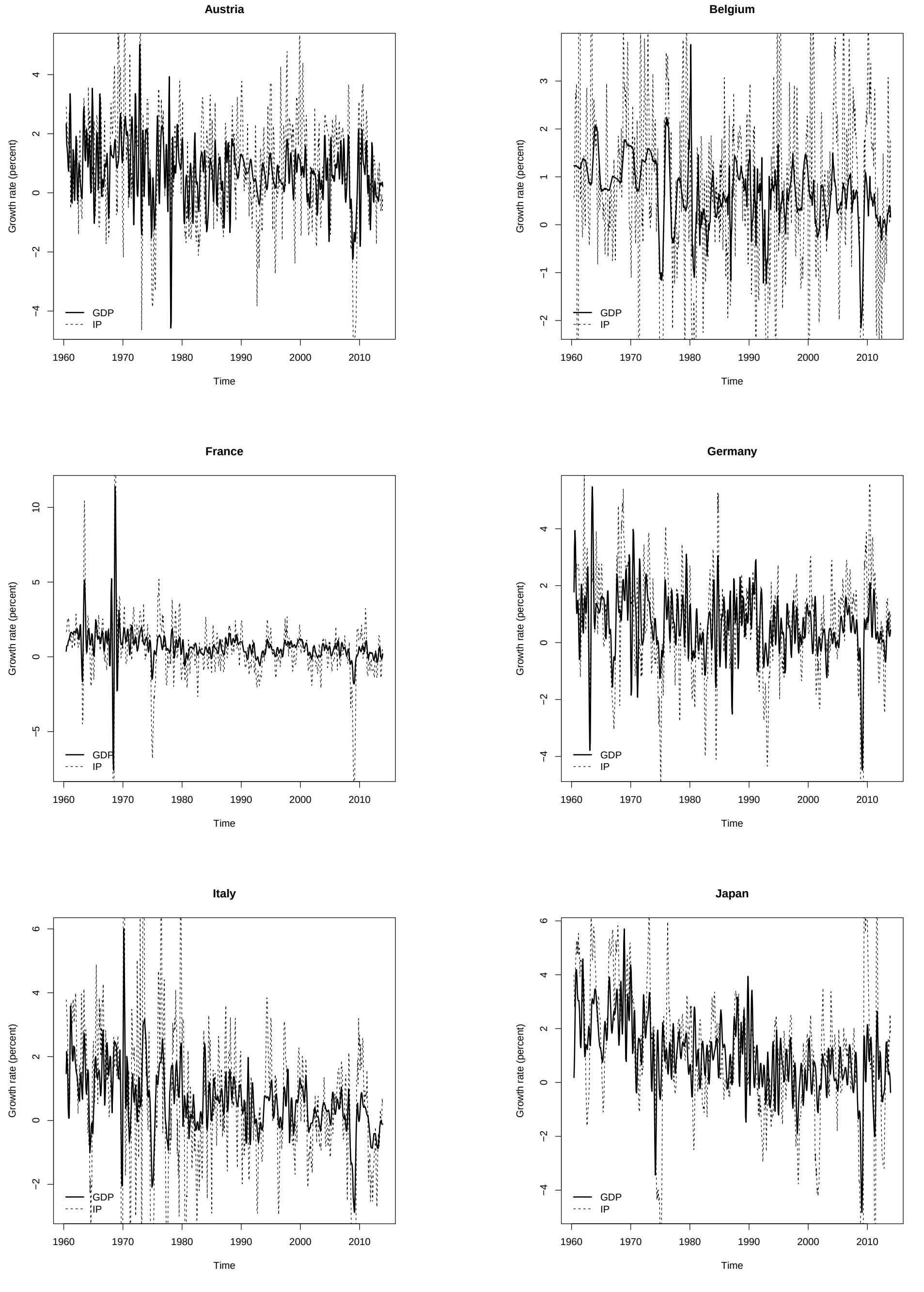

FIGURE 1: Monthly GDP and IP growth.

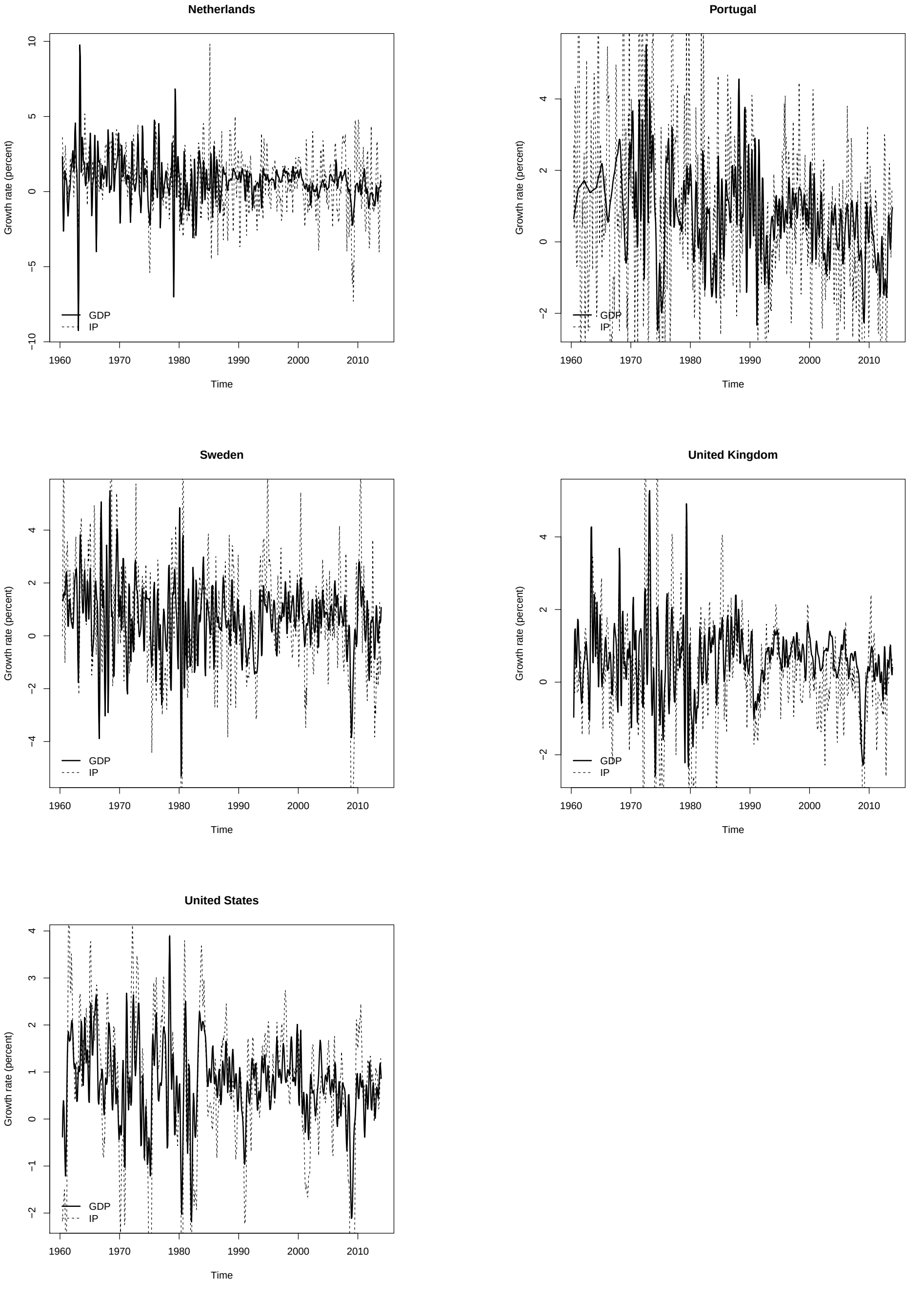

FIGURE 1: Monthly GDP and IP growth (continued).

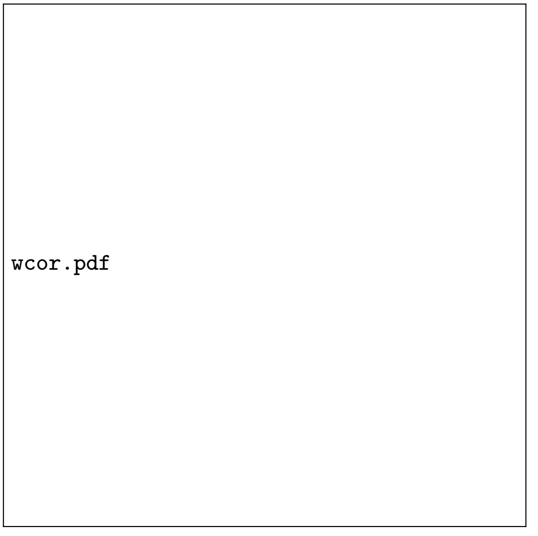

Figure 2: Weighted correlation for the US.

Table 2 also reports the correlations of the MFMSSA estimates with those obtained by using the widely used Chow-Lin method to disaggregate time series. The correlations range from 0.62 for the Netherlands up to 0.97 for Belgium, reaching more than 0.8 on average across all countries. As the Chow-Lin method, in its original formulation, applies only to static models, we also consider the dynamic model variant of the Chow-Lin method (see Santos Silva & Cardoso, 2001), as it is a more natural working assumption when dealing with time series. In this case, the MFMSSA estimates present even higher correlations for most countries, ranging from 0.72 for Sweden up to 0.99 for Belgium and achieving a value close to 0.9 on average across all countries. Such a comparison reinforces the plausibility of the monthly GDP estimates obtained using the MFMSSA approach.

Another exercise for evaluating the MFMSSA approach consists of assessing its usefulness in a forecasting context. In this case, one can compare the three-step-ahead monthly forecast of the interpolated series with the one-step-ahead quarterly forecast of the actual series. In particular, we take the full sample of observations, perform the MFMSSA interpolation and then take the resulting monthly estimates for GDP growth and forecast it three months ahead via a parsimonious AR model with the order length chosen according to standard information criteria. The forecasts obtained for the third month of each quarter can be compared directly with the ones obtained by simply fitting an AR model to the quarterly GDP series and forecasting one quarter ahead. This exercise allows us to assess how much is gained in terms of forecasting performance by

	IP	Chow-Lin	Dynamic model
Austria	0.48	0.69	0.90
Belgium	0.46	0.97	0.99
France	0.80	0.85	0.85
Germany	0.64	0.78	0.94
Italy	0.76	0.86	0.97
Japan	0.64	0.93	0.78
Netherlands	0.32	0.62	0.83
Portugal	0.47	0.91	0.96
Sweden	0.55	0.72	0.72
United Kingdom	0.63	0.77	0.77
United States	0.76	0.85	0.97

TABLE 2. Correlation of the MFMSSA monthly GDP with the monthly IP, the monthly GDP with the Chow-Lin method, and the monthly GDP with the dynamic extension of the Chow-Lin method.

taking on board the monthly dynamics obtained via MFMSSA.[3] We conduct the forecasting exercise using an expanding window and recursive model selection and estimation.[4] Since the business cycle frequency range is defined in the literature typically as between two and eight years, we considered a starting period of sixteen years so as to encompass at least two complete business cycles. The forecasting results of such an exercise are presented in Table 3. In particular, we report the relative RMSFE, i.e., the ratio of the RMSFE of the monthly model to that of its quarterly counterpart, as well as the relative MAFE defined in a similar way. Furthermore, we computed the superior predictive ability (SPA) test proposed by Hansen (2005), to compare the performances of the two forecasting models. Both loss functions (the mean squared error and the mean absolute error) are considered, and therefore we report SPA_{MSE} and SPA_{MAE} respectively. The statistic reported in the table refers to the SPA p-value, where a low value signals that the benchmark is outperformed by the MFMSSA-based approach.

Table 3 shows clearly that taking on board the monthly dynamics of GDP estimates can improve the forecasting performance. In fact, both the RMSFE and the MAFE are lower in the case where the monthly model is used for forecasting one quarter ahead. This finding holds for all countries under study,

3. Since the quarterly GDP growth figures correspond to the values in the third month of the corresponding quarter in the monthly series, as described in Section 2.2, one is already mixing the quarterly and monthly information in a spirit similar to the MIDAS approach when modelling the monthly series. However, richer dynamics can be exploited, since a higher frequency series is being considered when fitting the model. In fact, we find that resorting to a MIDAS approach does not deliver better forecasting results.

4. We also conducted the forecasting exercise using a rolling window scheme, but the forecasting performance deteriorated. This may reflect the fact that an expanding window means that more information is taken on board as time goes by.

	$RMSFE$	$MAFE$	SPA_{MSE}	SPA_{MAE}
Austria	0.606	0.613	0.001	0.000
Belgium	0.728	0.825	0.049	0.033
France	0.615	0.619	0.000	0.000
Germany	0.460	0.495	0.000	0.000
Italy	0.822	0.792	0.000	0.000
Japan	0.482	0.484	0.000	0.000
Netherlands	0.560	0.594	0.002	0.000
Portugal	0.446	0.446	0.000	0.000
Sweden	0.579	0.592	0.000	0.000
United Kingdom	0.545	0.539	0.001	0.000
United States	0.542	0.528	0.000	0.000

Table 3. Forecast evaluation with full sample estimates.

and the gains across countries are quite noteworthy. The average improvement in terms of the RMSFE and MAFE is around 40%. The countries where these gains are the highest are Germany, Japan, Portugal, the United Kingdom and the United States, whereas the countries which present lower gains include Belgium and Italy. Based on the SPA test results, one can conclude that the MFMSSA-based approach outperforms the benchmark, at the usual statistical significance level, for all countries and both loss functions.

We assess what the forecasting performance would have been in a pseudo real time scenario by using the same recursive approach as before, performing the MFMSSA interpolation within such a window, and then forecasting three months ahead in the case of the monthly model and one quarter ahead based on the quarterly model. The results are displayed in Table 4.

	$RMSFE$	$MAFE$	SPA_{MSE}	SPA_{MAE}
Austria	0.996	0.996	0.435	0.446
Belgium	0.881	0.949	0.139	0.222
France	0.803	0.777	0.008	0.001
Germany	0.944	0.958	0.070	0.129
Italy	0.925	0.900	0.056	0.021
Japan	0.986	0.981	0.365	0.318
Netherlands	0.934	0.985	0.069	0.346
Portugal	0.975	0.973	0.128	0.162
Sweden	0.972	1.014	0.256	0.599
United Kingdom	0.957	0.995	0.159	0.440
United States	0.994	1.002	0.426	0.513

Table 4. Forecast evaluation with recursive sample estimates.

Table 4 shows that one improves the one-quarter-ahead forecasts even when taking into account the pseudo real-time computation of the GDP monthly estimates. However, the gains are much smaller in this case, being around 5% on average. One notable case is France, where the forecasting gains were around 20%. Naturally, the SPA test results are weaker, although there is statistical evidence of a forecasting improvement over the benchmark in a few cases. Furthermore, one should stress that the computation of monthly estimates enables one to compute forecasts every month, meaning that one is not restricted to forecasting only on a quarterly basis. This can be a valuable feature for tracking economic conditions in a real-life environment.

5. Conclusions

This paper has suggested a new approach based on SSA for coping with data sampled at different frequencies. In particular, we lay out the theoretical foundations and the rationale underlying such a mixed-frequency multivariate SSA approach and address some practical issues related to its implementation.

The empirical application considered two variables, namely GDP and industrial production. The former was sampled at a quarterly frequency, whereas the latter was monthly. Applying the MFMSSA approach, we obtain monthly estimates for GDP growth and assess the forecasting performances of both a monthly model and its quarterly counterpart. We analyze a set of eleven developed countries over the period running from the beginning of 1960 up to the end of 2013.

The results obtained are quite promising. We find that taking on board the monthly dynamics obtained via the MFMSSA method allows noteworthy forecasting gains to be achieved for all countries. Although the gains are lower in a pseudo real-time exercise, one should note that such an approach also enables one to deliver monthly forecasts, which can constitute a valuable feature for monitoring economic evolution in a real-life environment.

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